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USING TAYLOR MAPS WITH SYNCHROTRON RADIATION EFFECTS INCLUDED

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Abstract

Routinely, particle tracking in accelerators is done either by tracking element-by-element which is slow, or by using a transfer map that does not take into account radiation effects. Here we present a method for using Taylor maps that have radiation effects included. The mapping is divided into a radioactive part and a symplectic part. The radiative part produces the correct second order stochastic correlations between all phase-space dimensions. And the symplectic part is handled by partial map inversion, which eliminates non-symplectic effects due to the finite truncation of the Taylor series. This enables tracking simulations to use maps of lower order than what would otherwise be necessary leading to a speedup of the simulation.

INTRODUCTION

Particle tracking is an important and widely used simulation tool since it is the only technique that can accurately and reliably probe the nonlinear effects that can develop in particle beams over many turns [1].

Particle tracking generally is done either element-by-element or using maps which transport particles over many lattice elements. The element-by-element tracking is most reliable but slow, especially for large machines with sometimes tens of thousands of elements. Map tracking, which uses a set of truncated Taylor series expansions to represent large sections of the accelerator, can be orders of magnitude faster. However, such maps can introduce errors from truncating the power series, which are usually non-symplectic, and which disturb tracking especially at large amplitudes. Furthermore, such maps have historically not included radiation effects, which are often essential, especially in electron rings.

To partly remedy this, a map can be constructed which includes radiation effects. Denoting the orbital phase space coordinates with respect to some reference coordinates (generally the closed orbit) by ζ the map is written:

$$\vec{\zeta}^f = \vec{Z}\left(\vec{\zeta}^i\right) + S\,\vec{\xi} \tag{1}$$

Superscript i indicates initial coordinates at the beginning of the map, superscript f indicates final coordinates at the end of the map, and Z is a truncated Taylor series transport map at some order n_0 with radiation damping (the deterministic part of the radiation effect) included. It will be assumed that $\vec{Z}$ has no constant part: $\vec{Z}(\vec{0}) = \vec{0}$. In the above equation, $\vec{S}$ is a 6×6 matrix which represents the fluctuation (stochastic) radiation effect and $\vec{\xi}$ is a vector of six independent Gaussian distributed random numbers with unit sigma and zero mean. Here higher order terms in the stochastic fluctuations have been ignored.

Construction of maps of the form Eq. (1) have been implemented in the software package FPP/PTC [2, 3] and this code has been interfaced to the accelerator simulation code Bmad [4]. In this paper will be discussed how to avoid non-symplectic effects by partial map inversion.

For a truncated Taylor map, if $n_0 > 1$, by virtue of missing terms higher than n_0 in the expansion, there will be non-symplectic behavior that is more severe at larger amplitudes. One way to avoid this is to calculate $\vec{Z}$ at a large enough n_o so that the non-symplectic behavior at the maximum amplitude of tracked particles is not significant. This has the disadvantage in that it increases computation time, both in computing the map initially and during tracking when the map must be repeatedly evaluated. Another possibility is to add terms to $\vec{Z}$ of order higher than n_0 to counteract the non-symplectic effects. Unfortunately, adding terms may lead to nonphysical behavior. For example, if stochastic radiation effects are included in the simulation, one of the authors (P. Nishikawa) has observed vertical beam blow up due to slight nonphysical non-linear anti-damping.

Presented in this paper is a third way to remove non-symplectic behavior. This involves "symplectifying the map". This is done by first working on the jets via "symplectic restoration" and then tracking is done via "symplectification" using a generating function on the nonlinear symplectic part of the map.

SYMPLECTIC RESTORATION

Symplectic restoration involves a rewriting of the jet $\vec{Z}$ as a concatenation of 4 jets, dropping the cumbersome vector arrows:

$$Z = L_r \circ N_r \circ L_s \circ N_s \tag{2}$$

L_r is a linear map near the identity which is nonsymplectic since it contains the effects of the radiation, N_r is a purely nonlinear jet which contains effects of the radiation, L_s is a linear symplectic map, and lastly N_s is a purely nonlinear symplectic jet, the linear part of which is the identity.

In the absence of radiation, the map Z is simply the symplectic map $L_s \circ N_s$. To get this factorization, we first extract the linear part of Z denoted Z_1. This map is almost symplectic since it is assumed that the radiation effect is small. A contraction mapping due to Furman (see [5], p. 5351, denoted S_1) is used to produce a symplectic map near Z_1:

$$L_s = \lim_{k\to\infty} \underbrace{S_1\left(\,S_1\left(\,S_1\left(\,\cdots Z_1 \cdots\right)\right)\right)}_{k\ times} = S_1^k\left(Z_1\right) \tag{3}$$

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For the map S_1 we choose:

$$S_1(M) = \frac{1}{2}\left[3I - MJM^\top J^\top\right] M \tag{4}$$

where the superscript $\top$ denotes the transpose and J is the matrix defining the Poisson bracket, that is, a rotation of 90° in each degree of freedom. In practice, L_s is calculated by applying S_1 until convergence is achieved.

Once L_s is computed, L_r can be evaluated using the equation

$$L_r = Z_1 \circ L_s^{-1} \tag{5}$$

The next step consists in something called "symplectic restoration" in reference [5] page 5347. We start with the fact that a jet Y near the identity can be written in the form:

$$Y = \exp\left(F \cdot \nabla\right) I \tag{6}$$

where F is some vector function. Using Eq. (2), the nonlinear part of Z is isolated:

$$M \equiv L_r^{-1} \circ Z \circ L_s^{-1} = N_r \circ L_s \circ N_s \circ L_s^{-1} \equiv N_r \circ \tilde{N}_s \tag{7}$$

We need to compute $\tilde{N}_s \equiv L_s \circ N_s \circ L_s^{-1}$ by extracting a Poisson bracket operator out of M. This technique is called symplectic restoration since the deviation from symplecticity is often due to integration inaccuracy rather than radiation. Here it is not the case since the deviations are due to radiation. If we assume that M can be written in the form

$$M = \exp\left(F \cdot \nabla\right) I \tag{8}$$

then it follows that

$$F \equiv \log\left(M\right) \tag{9}$$

The symbol $\equiv$ is used to emphasize that there is a big pile of notation abuses in Eq. (9). This logarithm procedure is described an appendix of reference [6]. It is not possible to define the logarithm of a nonlinear map of phase space or of its jet representation directly. Rather the map whose logarithm we take is the Lie map $\mathcal{M}$ of Dragt which is a map acting on functions:

$$F \cdot \nabla = \log\left(\mathcal{M}\right) \tag{10}$$

In jet space, $\mathcal{M}$ is represented by a matrix where the basis vectors are monomials of the phase space coordinates. This matrix[1] is denoted as $\mathbb{M}^\top$. See [8], page 113 for an example using the map of Eq. (2.1) of reference [8], or section 2.3.1 of [6].

Since this matrix is near the identity, the expansion for the logarithm convergences. In fact, since it is a nonlinear jet, it convergences in a finite number of steps because the jet minus the identity is nilpotent:

$$\log\left(M\right) = \sum_{n=0}^{n_o} \frac{(-1)^{n+1}}{n} \left(\mathcal{M} - \mathcal{I}\right)^n \tag{11}$$

[1] In the paper by Yu [7], even though the connection is not made explicitly, the matrix used by Yu (Eq. (1.5), is the matrix of Dragt's Lie map albeit transposed because Yu's matrix acts on vectors instead of components. The more natural map known by most people is the map acting on phase space moments which can be represented by a matrix if truncated. The matrix representation of the Lie map is the transpose of the matrix associated to moments.

This operation is central for the connection between tracking codes and Hamiltonian perturbation theory à la Guignard (Forest [6]).

The next step is to assume that the jet is mildly nonsymplectic and to extract a Poisson bracket operator $:w:$ out of F from which we will compute $\tilde{N}_s$. This procedure is mentioned in [5]:

$$w = \int_0^z JF(\tilde{z}) \cdot d\tilde{z} = \int_0^1 JF(\alpha z) \cdot z\, d\alpha \tag{12}$$

The first integral in Eq. (12) is over an arbitrary path. If the jet is symplectic, the result is path independent. Since we have radiation, the jet produced by $:w:$ is different from M. Now we use $:w:$ to compute the associated jet $\tilde{N}_{s;w}$:

$$\begin{aligned} \tilde{N}_{s;w} &= \exp\left(:w:\right) I \\ \Rightarrow N_s &= L_s^{-1} \circ \tilde{N}_{s;w} \circ L_s \end{aligned} \tag{13}$$

The radiative part, N_r, can be gotten using $\tilde{N}_{s;w}$:

$$N_r = M \circ \tilde{N}_{s;w}^{-1} \tag{14}$$

With this, Z is factored as advertised in Eq. (2).

EVALUATION OF THE DETERMINISTIC ORBITAL MAPS ON RAYS

We start with Eq. (2).

$$Z = L_r \circ N_r \circ L_s \circ N_s = W_r \circ L_s \circ N_s \tag{15}$$

Since the jet $W_r \equiv L_r \circ N_r$ is near the identity (the radiation effects are assumed small), to a good approximation, W_r can be computed and evaluated as a Taylor map.

The jet L_s is a linear symplectic matrix and thus a *bona fide* symplectic map on its own. We are left with the jet N_s which is a symplectic jet with only nonlinear terms. This jet is equivalent to the symplectic map generated by a generating function of mixed variables:

$$N_s \equiv G(q^f, p^i) \tag{16}$$

where

$$q^i = \frac{\partial G(q^f, p^i)}{\partial p^i}, \quad \text{and} \quad p^f = \frac{\partial G(q^f, p^i)}{\partial q^f} \tag{17}$$

This map is obtained via partial inversion of the jet N_s using a method due to M. Berz. The numerical evaluation involves a Newton search on Eq. (17).

Also it is important **not** to include the linear part L_s in the function G since it would make G become potentially arbitrarily.

EQUIVALENT SIMPLER PROCEDURE

We can simply factor the jet Z in terms of a linear part and a non-linear jet:

$$Z = L \circ N \tag{18}$$

The map L is linear and can be evaluated as is on a ray. The jet N cannot be evaluated as is since truncation induces gross violation of the symplectic condition. However we

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can partially invert it order by order and write an equation similar to Eq. (17).

$$N(q^i, p^i) \equiv N^q(q^f, p^i) \tag{19}$$

where

$$q^i_k = N^q_{2k-1}(q^f, p^i), \quad \text{and} \quad p^f_k = N^q_{2k}(q^f, p^i) \tag{20}$$

Here k runs from 1 to 3 representing the 3 positions and 3 momenta. The map N^q is partially inverted in the position variables. The jets N and N^q are not the same but they produce the same jets when properly interpreted via Eq. (20) and therefore the symbol $\equiv$ was used. However, using N^q with a Newton search for Eq. (20) produces a symplectic map under zero radiation conditions as opposed to tracking with N which will not, in general, be symplectic.

This is a simple alternate way to track a map without having to factorize it.

PREPARATION OF THE STOCHASTIC CONTRIBUTION TO TRACKING

Beam envelope theory incorporates the fluctuation of the quadratic moments of phase space variables into the map. For example, in the case of linear transport of phase pace coordinates by the matrix M, the second order moments matrix Σ is transported as:

$$\Sigma^f = M\,\Sigma^i M^\top + \Xi \quad \text{where} \quad \Sigma_{ij} = \langle z_i z_j \rangle \ . \tag{21}$$

The stochastic changes due to synchrotron radiation are captures in the matrix Ξ.

To use Eq. (21) we perform a Cholesky decomposition of the stochastic matrix Ξ:

$$\text{if} \quad \Xi = S\,S^\top \quad \text{then} \quad z^f = z^i + S\,\xi \tag{22}$$

The vector ξ is made of six independent random numbers of variance one following the users favourite distribution. Often Gaussian distributions are used. One can check the software implementation for linear beam transport and ergodically compute the equilibrium beam sizes: they must agree with theory.

One can also compute nonlinear moments and they should agree with nonlinear theory. This is available in the FPP package.

TEST IN THE PRESENCE OF SIGNIFICANT NONLINEARITIES

The beam typically occupies a small region of phase space and therefore a linear theory, as in Eq. (21), correctly predicts the beam sizes.

Since radiation happens in the longitudinal plane, impervious to anything, we need to excite nonlinear coupling with the transverse plane to see a nonlinear beam size effect. In a normal machine, this requires coupling with the horizontal direction. To observe this, a simulation was done in a small lattice with the linear tunes slightly below the $\nu_x - 2\nu_t = 1$ resonance. The results are shown in Table 1 which shows elements of the beam size matrix Σ_{ij} as computed five different ways.

Particles were tracked for 100 million turns, element by element, with a stochastic kick at every bend. This is the "Exact" column in Table 1. The "Linear" column is based on Eq. (21) and shows significant differences from the "exact" results as can be expected due to the resonance.

Extending Eq. (21) to include nonlinearities, transport of the beam sigma matrix can be cast in the form

$$\Sigma^f = \mathbb{M}\,\Sigma^i + \Xi \tag{23}$$

where $\mathbb{M}$ is the transpose of the Lie map in the deterministic case (see [8]). $\mathbb{M}$ was computed using PTC code using approximately 9,000,000 synchrotron integrals. The result for the equilibrium distribution is the column labeled "8th order" in Table 1. The theory behind this is used in [9] where it is compared to the nonlinearly averaged Fokker-Planck equation. The full theory has not yet been published. Parenthetically, this type of calculation was done using a Fokker-Planck equation in [10].

The columns labeled "2" and "3"in Table 1 represents the results of using second and third order factorized maps as described in this paper. With the lattice used, tracking through the second order Taylor map was unstable but the third order map gave acceptable results.

Taylor maps must be used with caution. We do not discuss here examples with spin, but as pointed out in the footnote, they have to be used with even more caution since the depolarization can depend heavily on the number of maps used.[2]

[2] The correct inclusion of quaternions for simulating spin in the "9,000,000" integrals has not been worked out. The correct stochastic one-turn map is also elusive for quaternions and we are forced to use several maps at this point to obtain the correct polarization.

Table 1: 11, 23, 25, and 33 Elements of the Beam Size Matrix Σ in Equilibrium as Calculated from the Results for a Number of Different Tracking Algorithms

	Analytical		Tracking		
Σ_{ij}	Linear	8th order	"Exact"	"2"	"3"
11	$4.49 \cdot 10^{-7}$	$6.15 \cdot 10^{-7}$	$6.09 \cdot 10^{-7}$	$6.41 \cdot 10^{-7}$	$6.06 \cdot 10^{-7}$
23	$1.37 \cdot 10^{-11}$	$1.88 \cdot 10^{-11}$	$1.86 \cdot 10^{-10}$	$1.96 \cdot 10^{-11}$	$1.85 \cdot 10^{-11}$
25	$1.05 \cdot 10^{-13}$	$-1.67 \cdot 10^{-14}$	$-1.42 \cdot 10^{-14}$	$-7.00 \cdot 10^{-15}$	$-1.74 \cdot 10^{-14}$
33	$3.89 \cdot 10^{-12}$	$4.07 \cdot 10^{-12}$	$4.03 \cdot 10^{-12}$	$4.03 \cdot 10^{-12}$	$4.14 \cdot 10^{-12}$

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